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van Zanten, J.H.

published in
Stochastic Processes and Their Applications
2007

DOI (link to publisher)
10.1016/j.spa.2006.05.013

document version
Publisher's PDF, also known as Version of record

Available online at www.sciencedirect.com
ScienceDirect
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Stochastic Processes and their Applications 117 (2007) 57–70
stochastic processes and their applications
www.elsevier.com/locate/spa

When is a linear combination of independent fBm's equivalent to a single fBm?

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Received 4 November 2005; received in revised form 19 May 2006; accepted 26 May 2006
Available online 23 June 2006

Abstract

We study and answer the question posed in the title. The answer is derived from some new necessary and sufficient conditions for equivalence of Gaussian processes with stationary increments and recent frequency domain results for the fBm. The result shows in particular precisely in which cases the local almost sure behaviour of a linear combination of independent fBm's is the same as that of a multiple of a single fBm.

Keywords: Equivalence; Fractional Brownian motion; Frequency domain; Reproducing kernels; Stationary increments

1. Introduction

Since the nineteen sixties and seventies the study of equivalence of the laws of Gaussian processes has been well developed. Early papers include [17,11,13]. See also the treatments in [12,9,10]. Due to the availability of new mathematical techniques and also the interest in new examples, for instance processes related to the fractional Brownian motion (fBm), the topic has recently received considerable attention again. Recent contributions in the area include [2,3,1,18,19].

On an abstract level the question of equivalence is well understood, necessary and sufficient conditions for equivalence can for instance be formulated in terms of the (time domain) reproducing kernel Hilbert spaces associated with the Gaussian processes (cf. [13]). These general conditions are however rather abstract and often not directly useful in concrete cases. Only if the class of Gaussian processes is sufficiently restricted do more or less concrete

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doi:10.1016/j.spa.2006.05.013

equivalence results become available. Examples are the results in [12] and [9] on stationary processes and [1] on Volterra processes.

The present paper is largely motivated by an interesting equivalence result due to Cheridito [2]. He proved that if W is a standard Brownian motion and X is an independent fBm with Hurst parameter H, then, for a constant $a \neq 0$, W and $W + aX$ are (locally) equivalent if and only if $H > 3/4$. (In fact, he showed that for $H \in (0, 1/2) \cup (1/2, 3/4]$, the process $W + aX$ is not a semimartingale.) This raises the more general question: *When is a linear combination of independent fBm's equivalent to a multiple of a single fBm?* We study this question in the setting of Gaussian processes with stationary increments. Using a frequency domain approach we obtain necessary and sufficient conditions for equivalence which are general enough to deal with the question at hand, and, in combination with recent frequency domain results on the fBm of Dzhaparidze and van Zanten [7], concrete enough to give a complete answer to the question.

We will find that if $X = \sum a_k X^k$ is a linear combination of independent fBm's with increasing Hurst indices $H_1 < \cdots < H_n$, then X is locally equivalent to $a_1 X^1$ if $H_2 - H_1 > 1/4$. Conversely, if X is equivalent to a multiple of an fBm this must be $a_1 X^1$, and then necessarily $H_2 - H_1 > 1/4$. The consequences for instance for the almost sure behaviour of sample paths are obvious. In the case that $H_2 - H_1 > 1/4$ the laws of $(X_t)_{t \in [0,T]}$ and $(a_1 X^1_t)_{t \in [0,T]}$ have the same null sets, and hence for any $B \in \mathscr{B}(\mathbb{R}^{[0,T]})$ it holds that $\mathbb{P}((X_t)_{t \in [0,T]} \in B) = 1$ if and only if $\mathbb{P}((a_1 X^1_t)_{t \in [0,T]} \in B) = 1$. If $H_2 - H_1 \leq 1/4$ however, the local a.s. behaviour of X is *not* the same as that of any fBm.

The organization of the paper is as follows. In the next section we present the result that provides the answer to the central question of the paper. Its proof relies on a number of general equivalence results for Gaussian processes with stationary increments which are derived in Sections 4 and 5, after some preparations have been carried out in Section 3. The proofs of the main results have been collected in the final Section 6.

2. Equivalence to the fBm

If $X = (X_t)_{t \geq 0}$ is a mean square continuous, centered Gaussian process with stationary increments that starts from 0 (we call such processes Gaussian si-processes from now on), there exists a unique symmetric Borel measure μ on the line such that $\int (1 + \lambda^2)^{-1} \mu(\mathrm{d}\lambda) < \infty$ and

$$\mathbb{E} X_s X_t = \int_{\mathbb{R}} \frac{(\mathrm{e}^{\mathrm{i}\lambda s} - 1)(\mathrm{e}^{-\mathrm{i}\lambda t} - 1)}{\lambda^2} \, \mu(\mathrm{d}\lambda)$$

for all $s, t \geq 0$ (cf., e.g., [5]). This measure is called the *spectral measure* of the process. The spectral measure of an fBm with Hurst parameter $H \in (0, 1)$ is absolutely continuous with respect to the Lebesgue measure; its density is given by

$$f_H(\lambda) = c_H |\lambda|^{1-2H}, \qquad c_H = \frac{\sin(\pi H) \Gamma(1 + 2H)}{2\pi} \tag{2.1}$$

(see for instance [16]). If $X^1, \ldots, X^n$ are independent fBm's with Hurst parameters $H_1 < \cdots < H_n$ and $a_1, \ldots, a_n \in \mathbb{R} \setminus \{0\}$, then the linear combination $X = \sum a_k X^k$ is a Gaussian si-process with spectral density $f = \sum a_k^2 f_{H_k}$. Observe that the density f satisfies

$$f(\lambda) = a_1^2 f_{H_1}(\lambda) + a_2^2 c_{H_2} |\lambda|^{1-2H_2} (1 + O(1))$$

for $\lambda \to \infty$. In other words, we can write the linear combination of fBm's as the sum of a multiple of an fBm with Hurst parameter H_1 and a second, independent si-process whose spectral density behaves like a multiple of $\lambda \mapsto |\lambda|^{1-2H_2}$ near infinity. Theorem 2.1 below deals with the question of equivalence in such situations.

For a fixed $T > 0$ we call two stochastic processes $X = (X_t)_{t \geq 0}$ and $Y = (Y_t)_{t \geq 0}$ *equivalent on* $[0, T]$ if the distributions of $(X_t)_{t \in [0,T]}$ and $(Y_t)_{t \in [0,T]}$ are equivalent measures on the measurable space $(\mathbb{R}^{[0,T]}, \mathscr{B}(\mathbb{R}^{[0,T]}))$. We call them *locally equivalent* if they are equivalent on $[0, T]$ for every $T > 0$.

Theorem 2.1. *Let X be an fBm with Hurst parameter $H \in (0, 1)$ and let Y be a Gaussian si-process independent of X, with spectral density f that is continuous outside a bounded set and that satisfies $f(\lambda) = C|\lambda|^{1-2H'}(1 + O(1))$ as $\lambda \to \infty$, for some $C > 0$ and $H' \in (0, 1)$.*

(i) *If X and $X + Y$ are equivalent on $[0, T]$ for some $T > 0$, then $H - H' > 1/4$.*
(ii) *If $H - H' > 1/4$, then X and $X + Y$ are locally equivalent.*

We prove this theorem in Section 6, after we have derived some general results on the equivalence of Gaussian si-processes.

The following corollary provides the answer to the question raised in the title of the paper.

Corollary 2.2. *Let $X = \sum a_k X^k$, where $X^1, \ldots, X^n$ are independent fBm's with Hurst parameters $H_1 < \cdots < H_n$ and $a_1, \ldots, a_n \in \mathbb{R} \setminus \{0\}$.*

(i) *If X is equivalent to a multiple of an fBm on $[0, T]$ for some $T > 0$, then X is equivalent to $a_1 X^1$ and $H_2 - H_1 > 1/4$.*
(ii) *If $H_2 - H_1 > 1/4$ then X and $a_1 X^1$ are locally equivalent.*

In view of the remarks preceding Theorem 2.1 the second statement of the corollary follows from the second statement of the theorem. The first statement also follows from the theorem, in combination with some general observations regarding equivalence of si-processes. Details are given in Section 6.

Example 2.3 (*Mixed Fractional Brownian Motion*). Cheridito [2] studied the so-called *mixed fBm*, which is defined as $W + aX$, where W is a standard Brownian motion, X is an independent fBm with arbitrary Hurst parameter $H \in (0, 1)$ and $a \neq 0$. He proved that if $H > 3/4$, the mixed fBm is equivalent to W. For $H \in (0, 1/2) \cup (1/2, 3/4]$ he proved that the mixed fBm is not a semimartingale, which implies that it is not equivalent to W in that case.

These equivalence results also follow from Corollary 2.2 (but the stronger semimartingality result does not). In addition, the corollary shows that if $H < 1/4$, then $W + aX$ is equivalent to aX. For $H \in [1/4, 1/2) \cup (1/2, 3/4]$ the mixed fBm is not on any interval $[0, T]$ equivalent to a multiple of an fBm.

3. Preliminaries

The results of Section 2 are consequences of more general frequency domain results for equivalence of Gaussian si-processes. In this section we prepare the necessary concepts and notation.

3.1. Basic set-up

As was already mentioned, the term *Gaussian si-process* will always refer to a mean square continuous, centered Gaussian process with stationary increments, starting from 0. Since we are only interested in the laws of such processes it is no restriction to assume that the underlying probability space is the canonical one. Throughout this section, X will denote the canonical process on the canonical path space $(\Omega, \mathscr{F}) = (\mathbb{R}^{[0,\infty)}, \mathscr{B}(\mathbb{R}^{[0,\infty)}))$. We will consider two probability measures $\mathbb{P}_1$ and $\mathbb{P}_0$ on this measurable space such that under $\mathbb{P}_j$, the canonical process X is a Gaussian si-process with spectral measure μ_j. So μ_1 and μ_0 are symmetric Borel measures on the line such that for all $s, t \geq 0$ and $j \in \{0, 1\}$,

$$\mathbb{E}_{\mathbb{P}_j} X_s X_t = \int \hat{1}_s \overline{\hat{1}_t} \, \mathrm{d}\mu_j, \tag{3.1}$$

where $\hat{1}_t(\lambda) = (\exp(\mathrm{i}\lambda t) - 1)/(\mathrm{i}\lambda)$ is the Fourier transform of the indicator function $1_t = 1_{(0,t)}$ of the interval $(0, t)$. More generally, the Fourier transform of a function f is defined as

$$\hat{f}(\lambda) = \int_{\mathbb{R}} f(u) \mathrm{e}^{\mathrm{i}u\lambda} \, \mathrm{d}u.$$

If for $T > 0$ the probability measures $\mathbb{P}_1$ and $\mathbb{P}_0$ are equivalent on $\mathbb{R}^{[0,T]} \subseteq \mathbb{R}^{[0,\infty)}$ we say that the Gaussian si-processes with spectral measures μ_1 and μ_0 are *equivalent on* $[0, T]$. When this is true for every $T > 0$ we call the processes *locally equivalent*.

3.2. Spaces of linear and quadratic functionals

With the (canonical) process X we associate some useful spaces of functionals. By $L^2(\mathbb{P}_j)$, for $j = 0, 1$, we denote the space $L^2(\Omega, \mathscr{F}, \mathbb{P}_j)$ of all square integrable functionals of the process X. For $T \geq 0$ and $j = 0, 1$ we define $\mathscr{H}_T^{(1)}(\mathbb{P}_j)$ as the closure in $L^2(\mathbb{P}_j)$ of the linear span of the set $\{X_t : t \in [0, T]\}$. In short, $\mathscr{H}_T^{(1)}(\mathbb{P}_j)$ is the space of $\mathbb{P}_j$-square integrable linear functionals of $(X_t)_{t \in [0,T]}$. Let r_j be the covariance function of X under $\mathbb{P}_j$. Then, for $T \geq 0$ and $j = 0, 1$, $\mathscr{H}_T^{(2)}(\mathbb{P}_j)$ is defined as the closure in $L^2(\mathbb{P}_j)$ of the linear span of the set $\{X_s X_t - r_j(s, t) : s, t \in [0, T]\}$. This is the space of centered, quadratic functionals of the process X.

3.3. Spectral isometries

In view of the spectral representation (3.1) we introduce for $T \geq 0$ and $j = 0, 1$ the space $\mathscr{L}_T^{(1)}(\mu_j)$, defined as the closure in $L^2(\mu_j) = L^2(\mathbb{R}, \mathscr{B}(\mathbb{R}), \mu_j)$ of $\mathscr{L}_T^e$, the latter being the span of the set $\{\hat{1}_t : t \in [0, T]\}$. Clearly (3.1) implies that the linear map $\Phi_j^{(1)} : \mathscr{L}_T^e \to \mathscr{H}_T^{(1)}(\mathbb{P}_j)$, defined by $\Phi_j^{(1)}(\hat{1}_t) = X_t$, extends to an isometry

$$\Phi_j^{(1)} : \mathscr{L}_T^{(1)}(\mu_j) \to \mathscr{H}_T^{(1)}(\mathbb{P}_j).$$

Similarly, we can define a linear map

$$\Phi_j^{(2)} : \mathscr{L}_T^{(1)}(\mu_j) \otimes \mathscr{L}_T^{(1)}(\mu_j) \to \mathscr{H}_T^{(2)}(\mathbb{P}_j)$$

with the property that $\Phi_i^{(2)}(\hat{1}_s \otimes \hat{1}_t) = X_s X_t - r_j(s, t)$ and such that for every $\psi \in \mathscr{L}_T^{(1)}(\mu_j) \otimes \mathscr{L}_T^{(1)}(\mu_j)$ which is symmetric in the sense that $\psi(\omega, \lambda) = \overline{\psi(\lambda, \omega)}$, we have the isometry relation

$$\mathbb{E}_{\mathbb{P}_j} \left(\Phi_j^{(2)}(\psi) \right)^2 = 2 \|\psi\|^2_{L^2(\mu_j \otimes \mu_j)}.$$

Here the tensor product $\varphi \otimes \psi$ of $\varphi \in \mathscr{L}_T^{(1)}(\mu_j)$ and $\psi \in \mathscr{L}_T^{(1)}(\mu_k)$ is defined by $(\varphi \otimes \psi)(\omega, \lambda) = \varphi(\omega)\overline{\psi(\lambda)}$. The space $\mathscr{L}_T^{(1)}(\mu_j) \otimes \mathscr{L}_T^{(1)}(\mu_k)$ is the closure in $L^2(\mu_j \otimes \mu_k)$ of the span of the set of functions $\{\varphi \otimes \psi : \varphi, \psi \in \mathscr{L}_T^e\}$, where $\mu_j \otimes \mu_k$ denotes the usual product of the measures μ_j and μ_k.

The maps $\Phi_j^{(1)}$ and $\Phi_j^{(2)}$ are essentially the first and second order multiple Wiener integrals associated with the isonormal Gaussian process on the separable Hilbert space $\mathscr{L}_T^{(1)}(\mu_j)$; see for instance [15] for the general construction and properties (note however that in the present situation the Hilbert space under consideration is complex). See also [12] or [9], who consider the analogous construction for stationary Gaussian processes (the function $\hat{1}_t$ is then replaced by $\lambda \mapsto \exp(\mathrm{i}\lambda t)$).

3.4. Reproducing kernel Hilbert space structures

It is well known that for $T \geq 0$, the closure in $L^2(\mu_j)$ of the span of the set $\{\hat{1}_t : t \in [-T, T]\}$ is a RKHS of entire functions in the sense of de Branges [4]. This is explained in detail in Chapter 6 of [6]. Let K_T^j be the reproducing kernel of this space. Then it is easy to deduce that $\mathscr{L}_T^{(1)}(\mu_j)$ is a RKHS of entire functions as well and that

$$S_T^j(\omega, \lambda) = \mathrm{e}^{\frac{\mathrm{i}(\lambda-\omega)T}{2}} K_{T/2}^j(\omega, \lambda) \tag{3.2}$$

is the reproducing kernel of $\mathscr{L}_T^{(1)}(\mu_j)$.

The fact that $\mathscr{L}_T^{(1)}(\mu_j)$ is a space of entire functions means that every element $\psi \in \mathscr{L}_T^{(1)}(\mu_j)$ has a version that is the restriction to the real line of an entire function on the complex plane. From now on we will always consider this smooth version of elements of $\mathscr{L}_T^{(1)}(\mu_j)$. That S_T^j is the reproducing kernel of $\mathscr{L}_T^{(1)}(\mu_j)$ means that $S_T^j(\omega, \cdot) \in \mathscr{L}_T^{(1)}(\mu_j)$ for every $\omega \in \mathbb{R}$ and

$$\left\langle \psi, S_T^j(\omega, \cdot) \right\rangle_{L^2(\mu_j)} = \int_{\mathbb{R}} \psi(\lambda) \overline{S_T^j(\omega, \lambda)}\, \mu_j(\mathrm{d}\lambda) = \psi(\omega)$$

for every $\omega \in \mathbb{R}$ and $\psi \in \mathscr{L}_T^{(1)}(\mu)$.

Two consequences of the reproducing property that we use at several places below are firstly that the span of the set of functions $\{S_T^j(\omega, \cdot) : \omega \in \mathbb{R}\}$ is dense in $\mathscr{L}_T^{(1)}(\mu_j)$, and secondly that for $\psi \in L^2(\mu_j)$, the function

$$\omega \mapsto \left\langle \psi, S_T^j(\omega, \cdot) \right\rangle_{L^2(\mu_j)}$$

is the orthogonal projection of ψ on $\mathscr{L}_T^{(1)}(\mu_j)$. We denote this projection by $\pi_{\mathscr{L}_T^{(1)}(\mu_j)} \psi$.

It is easy to see that the space $\mathscr{L}_T^{(1)}(\mu_j) \otimes \mathscr{L}_T^{(1)}(\mu_k)$ is a RKHS as well, with reproducing kernel

$$((\omega_1, \lambda_1), (\omega_2, \lambda_2)) \mapsto S_T^j(\omega_1, \omega_2) \overline{S_T^k(\lambda_1, \lambda_2)}.$$

Indeed, for every $\varphi \in \mathscr{L}_T^{(1)}(\mu_j)$ and $\psi \in \mathscr{L}_T^{(1)}(\mu_k)$ and $\omega, \lambda \in \mathbb{R}$ we have

$$\begin{aligned}\left\langle \varphi \otimes \psi, S_T^j(\omega, \cdot) \otimes S_T^k(\lambda, \cdot) \right\rangle_{L^2(\mu_j \otimes \mu_k)} &= \langle \varphi, S_T^j(\omega, \cdot)\rangle_{L^2(\mu_j)} \overline{\langle \psi, S_T^k(\omega, \cdot)\rangle_{L^2(\mu_k)}} \\ &= (\varphi \otimes \psi)(\omega, \lambda).\end{aligned}$$

Since $\mathscr{L}_T^{(1)}(\mu_j) \otimes \mathscr{L}_T^{(1)}(\mu_j)$ is the closure in $L^2(\mu_j \otimes \mu_k)$ of the span of $\{\varphi \otimes \psi : \varphi, \psi \in \mathscr{L}_T^e\}$ this reproducing property extends to all of $\mathscr{L}_T^{(1)}(\mu_j) \otimes \mathscr{L}_T^{(1)}(\mu_k)$, by continuity of inner products.

4. Necessary and sufficient conditions for equivalence

In this section we derive spectral conditions for the equivalence of Gaussian si-processes. The results can be viewed as adaptations and extensions of known equivalence results for stationary processes that can be found for instance in [12]. A key difference, crucial for the application we have in mind, is the use of reproducing kernels.

We call two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on a vector space V equivalent if there exist two constants $c, C \in (0, \infty)$ such that $c\|v\|_1 \le \|v\|_2 \le C\|v\|_1$ for all $v \in V$. As usual the notation "$\propto$" means equality up to a multiplicative constant.

Theorem 4.1. *Two Gaussian si-processes with spectral measures μ_0 and μ_1 are equivalent on $[0, T]$ if and only if the $L^2(\mu_0)$-norm and the $L^2(\mu_1)$-norm on the space $\mathscr{L}_T^e$ are equivalent and the function*

$$(\omega, \lambda) \mapsto S_T^0(\omega, \lambda) - \int S_T^0(\omega, \gamma)\overline{S_T^0(\lambda, \gamma)}\, \mu_1(\mathrm{d}\gamma) \tag{4.1}$$

belongs to $\mathscr{L}_T^{(1)}(\mu_0) \otimes \mathscr{L}_T^{(1)}(\mu_0)$. In the case of equivalence the Radon–Nikodym derivative is given by

$$\frac{\mathrm{d}\mathbb{P}_1}{\mathrm{d}\mathbb{P}_0} \propto \exp\left(-\frac{1}{2}\Phi_0^{(2)}(S_T^1 - S_T^0)\right).$$

Proof. The starting point of the proof is Theorem 5 on p. 84 of [12]. This theorem deals with stationary processes, but it is straightforward to adapt it to our setting. After doing so one obtains that the processes are equivalent on $[0, T]$ if and only if the $L^2(\mu_0)$-norm and the $L^2(\mu_1)$-norm on the space $\mathscr{L}_T^e$ are equivalent and there exists a $\psi \in \mathscr{L}_T^1(\mu_0) \otimes \mathscr{L}_T^{(1)}(\mu_0)$ such that for all $s, t \ge 0$,

$$\left\langle \hat{1}_s, \hat{1}_t \right\rangle_{L^2(\mu_1)} = \left\langle \hat{1}_s, \hat{1}_t \right\rangle_{L^2(\mu_0)} - \left\langle \psi, \hat{1}_s \otimes \hat{1}_t \right\rangle_{L^2(\mu_0 \otimes \mu_0)}.$$

Now assume first that the processes are equivalent on $[0, T]$. Then by linearity, the equality in the display holds with $\hat{1}_s$ and $\hat{1}_t$ replaced by arbitrary elements $\varphi_1, \varphi_2 \in \mathscr{L}_T^e$. If the processes are equivalent then $\mathscr{L}_T^{(1)}(\mu_1) = \mathscr{L}_T^{(1)}(\mu_0)$ and the $L^2(\mu_1)$ and $L^2(\mu_0)$-norms are equivalent on this space. Using the continuity of inner products it is then easily shown that we have the equality

$$\langle \varphi_1, \varphi_2 \rangle_{L^2(\mu_1)} = \langle \varphi_1, \varphi_2 \rangle_{L^2(\mu_0)} - \langle \psi, \varphi_1 \otimes \varphi_2 \rangle_{L^2(\mu_0 \otimes \mu_0)} \tag{4.2}$$

for all $\varphi_1, \varphi_2 \in \mathscr{L}_T^{(1)}(\mu_1) = \mathscr{L}_T^{(1)}(\mu_0)$. Now take $\varphi_1 = S_T^0(\omega, \cdot)$ and $\varphi_2 = S_T^0(\lambda, \cdot)$ and use the reproducing properties to find that

$$\psi(\omega, \lambda) = S_T^0(\omega, \lambda) - \int S_T^0(\omega, \gamma)\overline{S_T^0(\lambda, \gamma)}\, \mu_1(\mathrm{d}\gamma).$$

Conversely, assume that the $L^2(\mu_0)$-norm and the $L^2(\mu_1)$-norm on the space $\mathscr{L}_T^e$ are equivalent and that the function ψ defined through the last display is an element of $\mathscr{L}_T^{(1)}(\mu_0) \otimes \mathscr{L}_T^{(1)}(\mu_0)$. Then $\psi(\omega, \lambda) = \left\langle \psi, S_T^0(\omega, \cdot) \otimes S_T^0(\lambda, \cdot)\right\rangle_{L^2(\mu_0 \otimes \mu_0)}$, which shows that (4.2) holds for all $\varphi_1, \varphi_2 \in \operatorname{span}\{S_T^0(\omega, \cdot) : \omega \in \mathbb{R}\}$. It follows from that reproducing property that this span is dense in $\mathscr{L}_T^{(1)}(\mu_0)$. Using also the equivalence of the $L^2(\mu_j)$-norms it is then easily deduced that (4.2) holds in fact for all $\varphi_1, \varphi_2 \in \mathscr{L}_T^{(1)}(\mu_1) = \mathscr{L}_T^{(1)}(\mu_0)$. In particular we can take $\varphi_1 = \hat{1}_s$ and $\varphi_2 = \hat{1}_t$. By the (adaptation of the) cited theorem of [12], it follows that the processes are equivalent on $[0, T]$.

A straightforward adaptation of Theorem 8 on p. 88 of [12] implies that if the processes are equivalent on $[0, T]$, there exists a function $\varphi \in \mathscr{L}_T^{(1)}(\mu_1) \otimes \mathscr{L}_T^{(1)}(\mu_0)$ such that for all $s, t \geq 0$

$$\left\langle \hat{1}_s, \hat{1}_t \right\rangle_{L^2(\mu_1)} = \left\langle \hat{1}_s, \hat{1}_t \right\rangle_{L^2(\mu_0)} - \left\langle \varphi, \hat{1}_s \otimes \hat{1}_t \right\rangle_{L^2(\mu_0 \otimes \mu_1)}$$

and the Radon–Nikodym derivative is given by

$$\frac{\mathrm{d}\mathbb{P}_1}{\mathrm{d}\mathbb{P}_0} \propto \exp\left(-\frac{1}{2}\Phi_0^{(2)}(\varphi)\right).$$

Reasoning as in the second paragraph of this proof one finds that

$$\langle \varphi_1, \varphi_2 \rangle_{L^2(\mu_1)} = \langle \varphi_1, \varphi_2 \rangle_{L^2(\mu_0)} - \langle \varphi, \varphi_1 \otimes \varphi_2 \rangle_{L^2(\mu_0 \otimes \mu_1)}$$

for all $\varphi_1 \in \mathscr{L}_T^{(1)}(\mu_0)$ and $\varphi_2 \in \mathscr{L}_T^{(1)}(\mu_1)$. By taking $\varphi_1 = S_T^0(\omega, \cdot)$ and $\varphi_2 = S_T^1(\lambda, \cdot)$ and using the reproducing properties we see that

$$\varphi(\omega, \lambda) = S_T^1(\omega, \lambda) - S_T^0(\omega, \lambda).$$

This completes the proof of the theorem. □

Remark 4.2. (i) The fact that the $L^2(\mu_j)$-norms on $\mathscr{L}_T^e$ are equivalent implies in particular that

$$0 < \liminf_{t \to 0} \frac{\|\hat{1}_t\|^2_{L^2(\mu_1)}}{\|\hat{1}_t\|^2_{L^2(\mu_0)}} \leq \limsup_{t \to 0} \frac{\|\hat{1}_t\|^2_{L^2(\mu_1)}}{\|\hat{1}_t\|^2_{L^2(\mu_0)}} < \infty.$$

But $v_j(t) = \|\hat{1}_t\|^2_{L^2(\mu_j)}$ is just the variance function of the process under $\mathbb{P}_j$. So for equivalence it is necessary that

$$0 < \liminf_{t \to 0} \frac{v_1(t)}{v_0(t)} \leq \limsup_{t \to 0} \frac{v_1(t)}{v_0(t)} < \infty. \tag{4.3}$$

(This shows for instance that two fBm's with different Hurst parameters are orthogonal on any non-empty time interval, the variance function of an fBm with Hurst parameter $H \in (0, 1)$ being given by $v(t) = t^{2H}$.)

(ii) Since $S_T^0(\cdot, \gamma) \otimes S_T^0(\cdot, \gamma) \in \mathscr{L}_T^{(1)}(\mu_0) \otimes \mathscr{L}_T^{(1)}(\mu_0)$ for every fixed γ, it follows from the form of (4.1) that if the function is well defined and belongs to $L^2(\mu_0 \otimes \mu_0)$, it automatically belongs to $\mathscr{L}_T^{(1)}(\mu_0) \otimes \mathscr{L}_T^{(1)}(\mu_0)$.

(iii) Observe that in order to verify equivalence, only the reproducing kernel associated with one of the spectral measures has to be known, while both are needed to find an expression for the Radon–Nikodym derivative. In general, reproducing kernels are difficult to compute. This explains why there exist many cases in which is it known that two processes are equivalent, but there is no explicit expression for the derivative.

Any function $\psi \in \mathscr{L}_T^{(1)}(\mu_0) \otimes \mathscr{L}_T^{(1)}(\mu_0)$ defines a Hilbert–Schmidt integral operator on $\mathscr{L}_T^{(1)}(\mu_0)$. We will denote this operator by ψ as well, so

$$\psi\varphi(\omega) = \int \psi(\omega, \lambda)\varphi(\lambda)\, \mu_0(\mathrm{d}\lambda).$$

The condition on the $L^2(\mu_0)$ and $L^2(\mu_1)$-norms in Theorem 4.1 can be replaced by a condition on the spectrum $\sigma(\psi)$ of integral operator associated with the function (4.1). This is sometimes convenient in applications. The theorem then takes the following form.

Theorem 4.3. *Two Gaussian si-processes with spectral measures μ_0 and μ_1 are equivalent on $[0, T]$ if and only if the function ψ defined by*

$$\psi(\omega, \lambda) = S_T^0(\omega, \lambda) - \int S_T^0(\omega, \gamma)\overline{S_T^0(\lambda, \gamma)}\, \mu_1(\mathrm{d}\gamma)$$

belongs to $\mathscr{L}_T^{(1)}(\mu_0) \otimes \mathscr{L}_T^{(1)}(\mu_0)$ and $1 \notin \sigma(\psi)$. In the case of equivalence the Radon–Nikodym derivative is given by

$$\frac{\mathrm{d}\mathbb{P}_1}{\mathrm{d}\mathbb{P}_0} \propto \exp\left(-\frac{1}{2}\Phi_0^{(2)}(S_T^1 - S_T^0)\right).$$

Proof. If $\psi \in \mathscr{L}_T^{(1)}(\mu_0) \otimes \mathscr{L}_T^{(1)}(\mu_0)$ then for all $\varphi_1, \varphi_2 \in \mathscr{L}_T^{(1)}(\mu_0)$ it holds that

$$\langle \psi, \varphi_1 \otimes \varphi_2 \rangle_{L^2(\mu_0 \otimes \mu_0)} = \langle \psi\varphi_2, \varphi_1 \rangle_{L^2(\mu_0)},$$

where on the right-hand side ψ denotes the integral operator on $\mathscr{L}_T^{(1)}(\mu_0)$ which has ψ as kernel. The proof of Theorem 4.1 shows that if the function ψ defined by (4.1) belongs to $\mathscr{L}_T^{(1)}(\mu_0) \otimes \mathscr{L}_T^{(1)}(\mu_0)$, then

$$\frac{\|\varphi\|^2_{L^2(\mu_1)}}{\|\varphi\|^2_{L^2(\mu_0)}} = 1 - \frac{\langle \psi\varphi, \varphi \rangle_{L^2(\mu_0)}}{\|\varphi\|^2_{L^2(\mu_0)}}$$

for all $\varphi \in \mathscr{S} = \operatorname{span}\{S_T^0(\omega, \cdot) : \omega \in \mathbb{R}\}$. Hence the spectrum $\sigma(\psi)$ of the operator $\psi : \mathscr{L}_T^{(1)}(\mu_0) \to \mathscr{L}_T^{(1)}(\mu_0)$ is contained in $(-\infty, 1]$ and the $L^2(\mu_0)$-norm and $L^2(\mu_1)$-norm on $\mathscr{S}$ are equivalent if and only $1 \notin \sigma(\psi)$. But $\mathscr{S}$ is dense in $\mathscr{L}_T^{(1)}(\mu_0)$, so if $1 \notin \sigma(\psi)$ the norms are in fact equivalent on $\mathscr{L}_T^{(1)}(\mu_0)$ and in particular on $\mathscr{L}_T^e$. Conversely, if the norms are equivalent on $\mathscr{L}_T^e$ then also on $\mathscr{L}_T^{(1)}(\mu_0)$ and hence on $\mathscr{S}$, so that $1 \notin \sigma(\psi)$. □

Remark 4.4. (i) If $\mu_1 \geq \mu_0$ the integral operator associated with ψ is negative definite, so the condition $1 \notin \sigma(\psi)$ is trivially satisfied.

(ii) The proof also shows that the condition of equivalence of the $L^2(\mu_j)$-norms on $\mathscr{L}_T^e$ in Theorem 4.1 can be replaced by the one-sided condition that there exists a finite constant C such that

$$\|\varphi\|_{L^2(\mu_0)} \leq C\|\varphi\|_{L^2(\mu_1)}$$

for all $\varphi \in \mathscr{L}_T^e$. The reversed inequality is automatic if (4.1) belongs to $\mathscr{L}_T^{(1)}(\mu_0) \otimes \mathscr{L}_T^{(1)}(\mu_0)$.

5. Some consequences of the equivalence theorems

In this section we discuss two consequences of the general equivalence theorems that we need for the proof of Theorem 2.1.

5.1. Only the large frequencies matter

We prove that changing the spectrum of a Gaussian si-process on a bounded segment of the line leads to an equivalent process. So the equivalence of Gaussian si-processes depends only on the behaviour of their spectral measures near infinity.

Theorem 5.1. *If the spectral measures of two Gaussian si-processes are equal outside a bounded set, the processes are locally equivalent.*

Proof. Let μ_0 be a spectral measure and for a bounded Borel set $I \subseteq \mathbb{R}$, let $\mu_1(\mathrm{d}\lambda) = 1_{\mathbb{R}\setminus I}(\lambda)\,\mu_0(\mathrm{d}\lambda)$. We first prove that in that case the Gaussian si-processes with spectral measures μ_0 and μ_1 are locally equivalent.

Fix $T > 0$. The function ψ appearing in Theorem 4.3 is in this case given by

$$\psi(\omega, \lambda) = \int_I S_T^0(\omega, \gamma)\overline{S_T^0(\lambda, \gamma)}\,\mu_0(\mathrm{d}\gamma).$$

By Jensen's inequality

$$|\psi(\omega, \lambda)|^2 \leq \mu_0(I) \int_I |S_T^0(\omega, \gamma)|^2 |S_T^0(\lambda, \gamma)|^2\,\mu_0(\mathrm{d}\gamma).$$

Using the reproducing property and the fact I is bounded, so that $\mu_0(I) < \infty$, it follows that

$$\begin{aligned}\|\psi\|^2_{L^2(\mu_0\otimes\mu_0)} &\leq \mu_0(I) \int_I |S_T^0(\gamma, \gamma)|^2\,\mu_0(\mathrm{d}\gamma) \\ &\leq (\mu_0(I))^2 \sup_{\gamma\in I} |S_T^0(\gamma, \gamma)|^2 < \infty,\end{aligned}$$

whence $\psi \in \mathscr{L}_T^{(1)}(\mu_0) \otimes \mathscr{L}_T^{(1)}(\mu_0)$.

To prove equivalence on $[0, T]$ it remains to show that $1 \notin \sigma(\psi)$. Observe that for $\varphi \in \mathscr{L}_T^{(1)}(\mu_0)$ it holds that

$$\begin{aligned}(\psi\varphi)(\lambda) &= \int \varphi(\omega)\psi(\omega, \lambda)\,\mu_0(\mathrm{d}\omega) \\ &= \int \varphi(\omega) \left(\int_I S_T^0(\omega, \gamma)\overline{S_T^0(\lambda, \gamma)}\,\mu_0(\mathrm{d}\gamma)\right) \mu_0(\mathrm{d}\omega) \\ &= \int_I \varphi(\gamma)\overline{S_T^0(\lambda, \gamma)}\,\mu_0(\mathrm{d}\gamma) \\ &= \pi_{\mathscr{L}_T^{(1)}(\mu_0)}(\varphi 1_I)\end{aligned}$$

(recall the notation introduced in Section 3.4). So if $\psi\varphi = \varphi$ then $\varphi = \pi_{\mathscr{L}_T^{(1)}(\mu_0)}(\varphi 1_I)$ and in particular $\|\varphi\|_{L^2(\mu_0)} \leq \|\varphi 1_I\|_{L^2(\mu_0)}$, which implies that φ vanishes outside I, and hence, since φ is entire, $\varphi = 0$ identically. This shows that 1 does not belong to the spectrum of ψ.

To complete the proof, say that two spectral measures μ_0 and μ_1 are equal outside the bounded Borel set I. Put $\mu(\mathrm{d}\lambda) = 1_{\mathbb{R}\setminus I}(\lambda)\,\mu_0(\mathrm{d}\lambda) = 1_{\mathbb{R}\setminus I}(\lambda)\,\mu_1(\mathrm{d}\lambda)$. Then by the first part of the proof, the processes with spectral measures μ_0 and μ_1 are locally equivalent with the process with spectral measure μ, and hence with each other. □

Remark 5.2. (i) A Gaussian process with a compactly supported spectral measure has infinitely often differentiable sample paths (at least in mean square sense). The theorem implies that the sum of a given si-processes and an independent processes with compactly supported spectral measure is locally equivalent with the original process. In short: by adding a smooth enough independent process the equivalence class of a given si-process does not change.

(ii) The theorem generalizes Theorem 14 on p. 101 of [12]. The latter deals with stationary processes with spectral densities satisfying a bound of the type $f(\lambda) \geq c(1+\lambda^2)^{-n}$.

5.2. Sufficient conditions on spectral densities

The conditions for equivalence given by Theorems 4.1 and 4.3 are very general, but in specific applications they may be difficult to handle directly, because it is often difficult to find explicit expressions for the reproducing kernels. If the spectral measures have densities with respect to the Lebesgue measure we can give sufficient conditions for equivalence in terms of the densities and the reproducing kernel on the diagonal.

Theorem 5.3. *Suppose that the spectral measures μ_0 and μ_1 have positive densities f_0 and f_1 with respect to the Lebesgue measure. If there exists a finite constant C such that $\|\varphi\|_{L^2(\mu_0)} \leq C\|\varphi\|_{L^2(\mu_1)}$ for all $\varphi \in \mathscr{L}_T^e$ and*

$$\int_c^\infty \left(\frac{f_1(\lambda)-f_0(\lambda)}{f_0(\lambda)}\right)^2 S_T^0(\lambda,\lambda) f_0(\lambda)\,\mathrm{d}\lambda < \infty$$

for some $c > 0$, the Gaussian si-processes with spectral measures μ_1 and μ_0 are locally equivalent.

Proof. In view of Theorem 5.1 we may alter f_1 on a bounded Borel subset of the line without consequences for the equivalence. In particular, we may assume that $f_1 = f_0$ on the interval $(-c, c)$. Then since $|S_T^0(\omega,\lambda)|^2 = \left|\langle S_T^0(\omega,\cdot), S_T^0(\lambda,\cdot)\rangle_{L^2(\mu_0)}\right|^2 \leq S_T^0(\omega,\omega)S_T^0(\lambda,\lambda)$, the integrability condition implies that

$$S_T^0(\lambda,\cdot)\frac{f_1-f_0}{f_0} \in L^2(\mu_0)$$

for every $\lambda \in \mathbb{R}$. The function ψ of Theorem 4.3 is in this case given by

$$\psi(\omega,\lambda) = \int S_T^0(\omega,\gamma)\overline{S_T^0(\lambda,\gamma)}(f_1-f_0)(\gamma)\,\mathrm{d}\gamma.$$

Hence, using the notation for orthogonal projections introduced in Section 3.4, we have that

$$\psi(\omega,\lambda) = \pi_{\mathscr{L}_T^{(1)}(\mu_0)}\left(S_T^0(\omega,\cdot)\frac{f_1-f_0}{f_0}\right)(\lambda).$$

It follows that

$$\int\int |\psi(\omega,\lambda)|^2\mu_0(\mathrm{d}\omega)\mu_0(\mathrm{d}\lambda) = \int \left\|\pi_{\mathscr{L}_T^{(1)}(\mu_0)}\left(S_T^0(\omega,\cdot)\frac{f_1-f_0}{f_0}\right)\right\|^2_{L^2(\mu_0)} \mu_0(\mathrm{d}\omega)$$

$$\begin{aligned}
&\leq \int \left\| S_T^0(\omega,\cdot)\frac{f_1-f_0}{f_0}\right\|^2_{L^2(\mu_0)} \mu_0(\mathrm{d}\omega)\\
&= \int \left(\int |S_T^0(\omega,\gamma)|^2 \left(\frac{f_1(\gamma)-f_0(\gamma)}{f_0(\gamma)}\right)^2 \mu_0(\mathrm{d}\gamma)\right)\mu_0(\mathrm{d}\omega)\\
&= \int \left(\frac{f_1(\gamma)-f_0(\gamma)}{f_0(\gamma)}\right)^2 S_T^0(\gamma,\gamma) f_0(\gamma)\,\mathrm{d}\gamma.
\end{aligned}$$

So the integrability assumption implies that $\psi \in L^2(\mu_0 \otimes \mu_0)$, proving the theorem. □

Remark 5.4. The theorem extends Theorem 17 on p. 104 of [12], which deals with stationary processes and which assumes certain bounds on the spectral densities. In the statement of the theorem of [12] it is claimed that the integrability condition is also necessary. This is a typographical error however; only sufficiency is proved and following the theorem it is remarked that the condition seems to be close to necessary.

6. Proofs for Section 2

6.1. Reproducing kernel for the fBm

The spectral measure μ_0 of the fBm with Hurst parameter $H \in (0,1)$ has density f_H given by (2.1). For this process the structure of the space $\mathscr{L}_T^{(1)}(\mu_0)$ has recently been studied in the papers [7,8]. In particular, we have an explicit expression for the reproducing kernel S_T^0, namely

$$\begin{aligned}
\frac{S_T^0(2\omega,2\lambda)}{S_T^0(0,0)} &= (2-2H)\Gamma^2(1-H)\left(\frac{T^2\omega\lambda}{4}\right)^H \mathrm{e}^{\mathrm{i}T(\lambda-\omega)}\\
&\quad\times \frac{J_{-H}(T\omega)J_{1-H}(T\lambda)-J_{1-H}(T\omega)J_{-H}(T\lambda)}{T(\lambda-\omega)}
\end{aligned}$$

for $\omega \neq \lambda$ and, on the diagonal,

$$\begin{aligned}
\frac{S_T^0(2\omega,2\omega)}{S_T^0(0,0)} &= (2-2H)\Gamma^2(1-H)\left(\frac{T\omega}{2}\right)^{2H}\\
&\quad\times\left(J^2_{1-H}(T\omega)+\frac{2H-1}{T\omega}J_{-H}(T\omega)J_{1-H}(T\omega)+J^2_{-H}(T\omega)\right).
\end{aligned}$$

Here J_ν denotes the Bessel function of the first kind of order ν and Γ is the gamma function.

The proof of Theorem 2.1 relies on some asymptotic properties of the reproducing kernel S_T^0. These are collected for convenience in the following lemma.

Lemma 6.1. (i) *For every $\omega \in \mathbb{R}\setminus\{0\}$ there exist constants C_1, C_2, C_3 such that*

$$\begin{aligned}
|S_T^0(\omega,\lambda)| \sim C_1|\lambda|^{H-3/2}\bigg| & C_2\sin\left(T\lambda+\frac{1}{2}H\pi-\frac{1}{4}\pi\right)\\
&+ C_3\cos\left(T\lambda+\frac{1}{2}H\pi-\frac{1}{4}\pi\right)+O\left(\frac{1}{|\lambda|}\right)\bigg|
\end{aligned}$$

as $|\lambda|\to\infty$. It holds that $C_1>0$, $C_2=0$ if and only ω is a zero of $J_{-H}(T\cdot)$ and $C_3=0$ if and only ω is a zero of $J_{1-H}(T\cdot)$.

(ii) *For every* $\omega \in \mathbb{R}$ *and every compact set* $I \subseteq \mathbb{R}$ *there exists a constant* $C > 0$ *such that*

$$\sup_{\alpha \in I} |S_T^0(\omega, \alpha + \beta)| \leq C|\beta|^{H-3/2}$$

for $|\beta|$ *large enough.*

(iii) *There exists a constant* $C > 0$ *such that*

$$S_T^0(\lambda, \lambda) = C|\lambda|^{2H-1}\left(1 + O\left(\frac{1}{|\lambda|}\right)\right)$$

as $|\lambda| \to \infty$.

Proof. The Bessel function J_ν has the asymptotic behaviour

$$\sqrt{\frac{\pi x}{2}} J_\nu(x) = \cos\left(x - \frac{1}{2}\nu\pi - \frac{1}{4}\pi\right) + O\left(\frac{1}{|x|}\right)$$

for $|x| \to \infty$, cf. [14], p. 122. Together with the explicit expressions for the reproducing kernel and some straightforward computations this yields the statements of the lemma. □

6.2. *Proof of Theorem* 2.1(i)

Let μ_0 be the spectral measure of the fBm with Hurst parameter H and $\mu_1(\mathrm{d}\lambda) = \mu_0(\mathrm{d}\lambda) + f(\lambda)\,\mathrm{d}\lambda$. By Theorem 5.1 we may alter f on a bounded set, so we may assume that f is continuous on $\mathbb{R} \setminus (-1/2, 1/2)$, say.

By Theorem 4.1, the equivalence of the processes implies in particular that the $L^2(\mu_0)$ and $L^2(\mu_1)$-norms on $\mathscr{L}_T^{(1)}(\mu_0)$ are equivalent. It follows that for every $\varphi \in \mathscr{L}_T^{(1)}(\mu_0)$,

$$\int |\varphi(\lambda)|^2 f(\lambda)\,\mathrm{d}\lambda \leq \|\varphi\|^2_{L^2(\mu_1)} \lesssim \|\varphi\|^2_{L^2(\mu_0)} < \infty.$$

In particular we have, for every $\omega \in \mathbb{R}$,

$$\int |S_T^0(\omega, \lambda)|^2 f(\lambda)\,\mathrm{d}\lambda < \infty.$$

In view of part (i) of Lemma 6.1 this implies that $H - H' < 1/2$.

The function ψ of Theorem 4.3 is in this case given by

$$\psi(\omega, \lambda) = \int S_T^0(\omega, \gamma)\overline{S_T^0(\lambda, \gamma)} f(\gamma)\,\mathrm{d}\gamma.$$

By Fubini's theorem and a change of variables we have

$$\begin{aligned} |\psi(\omega, \lambda)|^2 &= \int\int S_T^0(\omega, \gamma)\overline{S_T^0(\lambda, \gamma)} S_T^0(\lambda, \gamma')\overline{S_T^0(\omega, \gamma')} f(\gamma) f(\gamma')\,\mathrm{d}\gamma\mathrm{d}\gamma' \\ &= \int\int S_T^0(\omega, \alpha + \beta)\overline{S_T^0(\lambda, \alpha + \beta)} S_T^0(\lambda, \beta)\overline{S_T^0(\omega, \beta)} f(\alpha + \beta) f(\beta)\,\mathrm{d}\alpha\mathrm{d}\beta. \end{aligned}$$

We write the second integral as the sum of the integral over the set $A_\varepsilon = \{(\alpha, \beta) : |\alpha| \leq \varepsilon, |\beta| \geq 1\}$ and the integral over its complement A_ε^c. The integral over A_ε^c converges to $|\psi(\omega, \lambda)|^2$ as $\varepsilon \to 0$, so in particular it is non-negative for small enough ε. For the other integral we have

$$\begin{aligned}
&\frac{1}{\varepsilon}\int\int_{A_\varepsilon} S_T^0(\omega,\alpha+\beta)\overline{S_T^0(\lambda,\alpha+\beta)}S_T^0(\lambda,\beta)\overline{S_T^0(\omega,\beta)}f(\alpha+\beta)f(\beta)\,\mathrm{d}\alpha\mathrm{d}\beta \\
&\quad=\int_{|\beta|\geq 1} h_\varepsilon(\beta)\,\mathrm{d}\beta,
\end{aligned}$$

where

$$h_\varepsilon(\beta)=\left(\frac{1}{\varepsilon}\int_{|\alpha|\leq\varepsilon} S_T^0(\omega,\alpha+\beta)\overline{S_T^0(\lambda,\alpha+\beta)}f(\alpha+\beta)\,\mathrm{d}\alpha\right)S_T^0(\lambda,\beta)\overline{S_T^0(\omega,\beta)}f(\beta).$$

By continuity of the last integrand, $h_\varepsilon(\beta)\to|S_T^0(\omega,\beta)|^2|S_T^0(\lambda,\beta)|^2f^2(\beta)$ as $\varepsilon\to 0$. Moreover, by part (ii) of Lemma 6.1,

$$\begin{aligned}
|h_\varepsilon(\beta)| &\leq |S_T^0(\lambda,\beta)||S_T^0(\omega,\beta)|f(\beta)\sup_{|\alpha|\leq\varepsilon}|S_T^0(\omega,\alpha+\beta)||S_T^0(\lambda,\alpha+\beta)|f(\alpha+\beta) \\
&\leq C\beta^{4(H-H')-4}
\end{aligned}$$

for ε small enough and large enough β. We just showed that $H-H'<1/2$, so the exponent is less than -2, and hence the functions h_ε are dominated by an integrable function. By dominated convergence it follows that

$$\frac{1}{\varepsilon}\int\int_{A_\varepsilon}\to\int_{|\beta|\geq 1}|S_T^0(\omega,\beta)|^2|S_T^0(\lambda,\beta)|^2f^2(\beta)\,\mathrm{d}\beta$$

and the integral on the right-hand side is finite. The integral is also strictly positive, since the functions $S_T^0(\omega,\cdot)$ are non-vanishing entire functions. Hence, for $\varepsilon>0$ small enough,

$$\frac{1}{\varepsilon}\int\int_{A_\varepsilon}\geq\frac{1}{2}\int_{|\beta|\geq 1}|S_T^0(\omega,\beta)|^2|S_T^0(\lambda,\beta)|^2f^2(\beta)\,\mathrm{d}\beta.$$

So we see that for $\varepsilon>0$ small enough,

$$|\psi(\omega,\lambda)|^2\geq\frac{\varepsilon}{2}\int_{|\beta|\geq 1}|S_T^0(\omega,\beta)|^2|S_T^0(\lambda,\beta)|^2f^2(\beta)\,\mathrm{d}\beta.$$

By Theorem 4.3 equivalence implies that $\psi\in L^2(\mu_0\otimes\mu_0)$. Integrating the inequality and using the fact that

$$\int|S_T^0(\omega,\beta)|^2\,\mu_0(\mathrm{d}\omega)=S_T^0(\beta,\beta),$$

we find that equivalence of the processes implies that

$$\int_{|\beta|\geq 1}|S_T^0(\beta,\beta)|^2f^2(\beta)\,\mathrm{d}\beta<\infty.$$

In view of part (iii) of Lemma 6.1 we must then have $H'-H>1/4$.

6.3. *Proof of Theorem* 2.1(ii)

Follows from Theorem 5.3 and part (iii) of Lemma 6.1.

6.4. Proof of Corollary 2.2(i)

If the process X with spectral density f is equivalent on $[0, T]$ (with $T > 0$) to a multiple of an fBm with Hurst parameter H, then by Theorem 4.1 the $L^2(f)$ and $L^2(f_H)$-norms must be equivalent on $\mathscr{L}_T^e$. In particular,

$$0 < \liminf_{t\to 0} \frac{\|\hat{1}_t\|^2_{L^2(f)}}{\|\hat{1}_t\|^2_{L^2(f_H)}} \leq \limsup_{t\to 0} \frac{\|\hat{1}_t\|^2_{L^2(f)}}{\|\hat{1}_t\|^2_{L^2(f_H)}} < \infty$$

(see Remark 4.2). Since $\|\hat{1}_t\|^2_{L^2(f)} = \sum a_k^2 t^{2H_k}$ and $\|\hat{1}_t\|^2_{L^2(f_H)} = t^{2H}$, we must then have $H = H_1$. Hence, X is equivalent $a_1 X^1$ on $[0, T]$ in that case. The rest of the statement follows from Theorem 2.1.

Acknowledgements

This research was partially funded by the Netherlands Organization for Scientific Research (NWO). Thanks to Tommi Sottinen for his comments on an earlier version of this paper and to Esko Valkeila for having given me the opportunity to present part of this work at a workshop in Jyväskylä in the spring of 2005.

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